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Optical Resolution Through a Randomly Inhomogeneous Medium for Very Long and Very Short Exposures

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A theoretical foundation is developed for relating the statistics of wave distortion to optical resolution. The average resolution of very-long- and very-short-exposure images is studied in terms of the phase- and log-amplitude-structure functions, whose sum we call the "wave-structure function." Those results which are comparable are in agreement with the findings of Hufnagel and Stanley who studied the average modulation transfer function of long-exposure images. It is found that the average short-exposure resolution can be significantly better than the average long-exposure resolution, but only if the wave distortion does not include substantial intensity variation.

INDEX HEADINGS: Atmospheric optics; Modulation transfer function; Resolving power; Image formation.

I. INTRODUCTION

ATMOSPHERIC turbulence with its associated random refractive-index inhomogeneities disturbs a light beam which propagates any significant distance through the atmosphere. The disturbance takes the form of distortion of the shape of the wavefront and variations of the intensity across the wavefront. If a collimated beam passes through the atmosphere and is then collected by a lens and brought to focus, the quality of the image formed is influenced by the atmospherically produced disturbances. That the distortion of the shape of the wavefront affects the image quality is obvious. It is not quite so obvious that the intensity variations across the wavefront affects the image quality. That this is so can be seen by considering the intensity variations as a form of random apodization of the lens. The apodization, of course, affects the image quality.

As an initial measure of image quality, we consider the modulation transfer function[1] (MTF) of an optical system composed of the atmosphere and a lens. We restrict our attention to thin, diffraction-limited lenses. In our study of the effect of wavefront distortion on the MTF, two distinct cases are to be considered. Some part of the distortion can be considered to be a random tilt of the wavefront. This tilt displaces the image but does not reduce its sharpness. If a very-short-exposure image is recorded, the image sharpness and the MTF are insensitive to the tilt, which can be a substantial part of the total distortion.[2] If a long-exposure image is recorded, the image is spread during the exposure by the random variations of the tilt. Hence, the image sharpness and the MTF are affected by wavefront tilt as well as by the more complex shapes. The analytic distinction between the two cases, which we refer to as the long-exposure case and the short-exposure case, lies in the manner in which the average of the MTF is taken. In the short-exposure case, a random factor associated with the tilt is extracted from the MTF before we take the average. In the long-exposure case, no such factor is removed.

II. THE UNAVERAGED MTF

Let us consider a spherical wave with radius of curvature R as formed by a plane wave immediately after passing through a thin diffraction-limited lens of focal

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[1] For the concept of MTF to be meaningful, a reasonable-size isoplanatism patch must exist, and we so assume. Though the MTF is conventionally defined in terms of the amplitude of the image of a unit-amplitude sine-wave test pattern in the patch, we work with the MTF defined in terms of the Fourier spectrum of the image response to a unit impulse in the isoplanatism patch. So long as the isoplanatism patch is large enough, there is, in effect, no difference in the definitions. For the unit impulse we use what is nominally an infinite plane wave, though we could equally well use a spherical wave with an adjustment of the image plane, or any other wavefront that should produce a point image.

[2] D. L. Fried, J. Opt. Soc. Am. **55**, 1427 (1965); **56**, 410E (1966).

length R. Each point of the spherical wave may be associated with a corresponding point on a plane tangent to the lens and parallel to the plane wave. We denote the point by a vector $\mathbf{v}$ (with magnitude v) which is measured from an origin at the point of tangency of the plane and the lens. We let D be the diameter of the lens. If $U(\mathbf{v})$ is the complex quantity which describes a wave that deviates (not grossly) in amplitude and phase from a spherical wave (in the same manner as the wave collected by the lens deviates from a plane wave), then it is well known[3] that $u(\mathbf{x})$, the complex quantity which describes the phase and amplitude of the image at a point $\mathbf{x}$ in the focal plane of the lens, is given to a good approximation by

$$u(\mathbf{x})=A\int d\mathbf{v}U(\mathbf{v})\exp\left(-i\frac{2\pi}{\lambda R}\mathbf{v}\cdot\mathbf{x}\right), \tag{2.1}$$

where A is a normalization constant and λ is the wavelength of the light. $U(\mathbf{v})$ and $u(\mathbf{x})$, though described simply as complex quantities which specify spatial phase and amplitude, are the electric vectors (without the high-frequency time-dependence.) The origin in the focal plane, i.e., $\mathbf{x}=0$ is the perpendicular projection (perpendicular relative to the $\mathbf{v}$ plane) of the origin in the $\mathbf{v}$ plane upon the $\mathbf{x}$ plane. The integration is over the infinite $\mathbf{v}$ plane but $U(\mathbf{v})=0$ outside the area of the lens, i.e., when $2v>D$.

The intensity of the image is $u^*(\mathbf{x})u(\mathbf{x})$. The MTF of the image-forming optical system,[4] which is the normalized two-dimensional (spatial) Fourier transform of the intensity of the image, can be written as

$$\tau(\mathbf{f})=B\int d\mathbf{x}u^*(\mathbf{x})u(\mathbf{x})\exp(2\pi i\mathbf{f}\cdot\mathbf{x}), \tag{2.2}$$

where B is a normalization constant, chosen so that

$$\tau(0)=1, \tag{2.3}$$

and $\mathbf{f}$ is a spatial frequency vector, whose magnitude is f. At various times, where appropriate, we switch from $\tau(\mathbf{f})$ to $\tau(f)$ without comment. The meaning of the latter should be obvious in each case.

By substituting (2.1) into (2.2) we get

$$\tau(\mathbf{f})=A^2B\int\int\int d\mathbf{x}d\mathbf{v}d\mathbf{v}'U^*(\mathbf{v}')U(\mathbf{v}) \times\exp[2\pi i\mathbf{x}\cdot(\mathbf{f}+\mathbf{v}'/\lambda R-\mathbf{v}/\lambda R)]. \tag{2.4}$$

By performing the $\mathbf{x}$-integration, we obtain a delta function which makes performance of the $\mathbf{v}'$-integration trivial. After performing the $\mathbf{v}'$-integration we have

$$\tau(\mathbf{f})=A^2B\int d\mathbf{v}U^*(\mathbf{v}-\lambda R\mathbf{f})U(\mathbf{v}). \tag{2.5}$$

Now we restrict our attention to the case in which the lens is diffraction-limited but the wave collected by the lens deviates from a uniform-amplitude plane wave because of random phase and amplitude perturbations. We let $\phi(\mathbf{v})$ denote the random variable which describes the phase variation at the point $\mathbf{v}$, and let $l(\mathbf{v})$ denote the random perturbations of the logarithm of the amplitude. The zero references for measurement of $\phi(\mathbf{v})$ and $l(\mathbf{v})$ are chosen so that each vanishes when there is no perturbation. Without loss of generality, we consider the incoming wave to have unit amplitude when there is no perturbation. We can write

$$U(\mathbf{v})=W(\mathbf{v})\exp[l(\mathbf{v})+i\phi(\mathbf{v})], \tag{2.6}$$

where $W(\mathbf{v})$ is an "aperture function" defined as

$$W(\mathbf{v})=\begin{cases}1 & \text{if } v\leqslant D/2\\ 0 & \text{if } v>D/2.\end{cases} \tag{2.7}$$

We now rewrite (2.5) as

$$\tau(\mathbf{f})=A^2B\int d\mathbf{v}(W\mathbf{v}-\lambda R\mathbf{f})W(\mathbf{v}) \times\exp\{[(\mathbf{v})+l(\mathbf{v}-\lambda R\mathbf{f})]+i[\phi(\mathbf{v})-\phi(\mathbf{v}-\lambda R\mathbf{f})]\}. \tag{2.8}$$

Since ϕ and l are random variables, so is τ. The average value of τ, studied in the next two sections, depends on how the average of (2.8) is taken. As indicated in Sec. 1, there is a dependence on the nature of the phase distortion and on the length of the exposure.

III. LONG-EXPOSURE MTF

For a long exposure, ϕ and l vary through a reasonable portion of the ensemble average of all the values they can possibly assume. Thus, for a single long-exposure image, the MTF has the value

$$\tau(\mathbf{f})=A^2B\int d\mathbf{v}W(\mathbf{v}-\lambda R\mathbf{f})W(\mathbf{v})\langle\exp\{[l(\mathbf{v})+l(\mathbf{v}-\lambda R\mathbf{f})] +i[\phi(\mathbf{v})-\phi(\mathbf{v}-\lambda R\mathbf{f})]\}\rangle, \tag{3.1}$$

where the angle brackets, $\langle\ \rangle$, are used to denote an ensemble average. The average long-exposure MTF, denoted by $\langle\tau(\mathbf{f})\rangle_{LE}$ is simply the ensemble average of the right-hand side of (3.1). If we commute the averaging and integrating operations, and note that the average of the average is simply the average, we see that

$$\langle\tau(\mathbf{f})\rangle_{LE}=A^2B\int d\mathbf{v}W(\mathbf{v}-\lambda R\mathbf{f})W(\mathbf{v}) \times\langle\exp\{[l(\mathbf{v})+l(\mathbf{v}-\lambda R\mathbf{f})]+i[\phi(\mathbf{v})-\phi(\mathbf{v}-\lambda R\mathbf{f})]\}\rangle. \tag{3.2}$$

[3] M. Born and E. Wolf, *Principles of Optics*, 2nd ed. (Pergamon Press Ltd., Oxford, 1964), p. 385, Equation (38).

[4] We consider the combination of turbulent atmosphere in the propagation path and the lens itself as the image-forming optical system.

Based on reasoning concerning the way the atmosphere produces phase and log-amplitude fluctuations, and by invoking the central limit theorem, it can be shown that $\phi(\mathbf{v})$ and $l(\mathbf{v})$ have gaussian distributions.[5] Further, we can show that the statistics of $\phi(\mathbf{v})$ and $l(\mathbf{v})$ are locally homogeneous and isotropic. As a consequence of isotropy, it follows that

$$\langle[\phi(\mathbf{v})-\phi(\mathbf{v}')][l(\mathbf{v})+l(\mathbf{v}')]\rangle=0, \qquad (3.3)$$

so that $[\phi(\mathbf{v})-\phi(\mathbf{v}-\lambda R\mathbf{f})]$ and $[l(\mathbf{v})+l(\mathbf{v}-\lambda R\mathbf{f})]$ may be considered to be independent gaussian random variables. Because the statistics of $\phi(\mathbf{v})$ are locally homogeneous, the mean of $[\phi(\mathbf{v})-\phi(\mathbf{v}-\lambda R\mathbf{f})]$ vanishes.

We denote the mean of $l(\mathbf{v})$ [and of $l(\mathbf{v}-\lambda R\mathbf{f})$] by $\bar{l}$, and use l for $l(\mathbf{v})$ when the value of $\mathbf{v}$ is of no consequence. The phase- and log-amplitude-structure functions, $\mathfrak{D}_\phi(r)$ and $\mathfrak{D}_l(r)$ are defined by

$$\mathfrak{D}_\phi(r)=\langle[\phi(\mathbf{v})-\phi(\mathbf{v}')]^2\rangle, \qquad (3.4a)$$

$$\mathfrak{D}_l(r)=\langle[l(\mathbf{v})-l(\mathbf{v}')]^2\rangle \qquad (3.4b)$$

where

$$r=|\mathbf{v}-\mathbf{v}'|. \qquad (3.5)$$

The log-amplitude covariance $C_l(r)$ is given by

$$C_l(r)=\langle[l(\mathbf{v})-\bar{l}][l(\mathbf{v})-\bar{l}]\rangle, \qquad (3.6)$$

from which it is easy to show that

$$\mathfrak{D}_l(r)=2[C_l(0)-C_l(r)]. \qquad (3.7)$$

We now need to evaluate $\bar{l}$. To carry this out, we now state the following result, which we use several times in the balance of this paper. It requires no more than a double integration over the appropriate probability distributions to show that if α and β are independent gaussian random variables, then

$$\langle\exp(a\alpha+b\beta)\rangle=\exp\{\tfrac{1}{2}[a^2\langle(\alpha-\bar{\alpha})^2\rangle+b^2\langle(\beta-\bar{\beta})^2\rangle]+(a\bar{\alpha}+b\bar{\beta})\}, \qquad (3.8)$$

where a and b are arbitrary complex coefficients and $\bar{\alpha}$ and $\bar{\beta}$ are the mean values of α and β.

To evaluate $\bar{l}$, we consider the interrelationship between $\bar{l}$ and $C_l(0)$ necessary to ensure conservation of energy. Consider $\exp(l)$, the random amplitude at some point of an infinite plane wave whose amplitude would be unity if there were no perturbations. The average intensity is

$$\langle I\rangle=\tfrac{1}{2}\langle e^{2l}\rangle. \qquad (3.9)$$

From energy-conservation considerations[6], we know that

$$\langle I\rangle=\tfrac{1}{2}. \qquad (3.10)$$

From (3.8) and (3.6), we see that

$$\langle e^{2l}\rangle=\exp[2C_l(0)+2\bar{l}]. \qquad (3.11)$$

The necessary and sufficient condition for (3.9), (3.10), and (3.11) to be self-consistent is that

$$\bar{l}=-C_l(0). \qquad (3.12)$$

The necessity of (3.12) for conservation of energy has been noted, without proof, by Chase.[7]

We are now ready to evaluate the ensemble average on the right-hand side of (3.2). Applying (3.8), (3.12), (3.6), (3.7), and (3.4a), we get

$$\begin{aligned}\langle\exp\{[l(\mathbf{v})+l(\mathbf{v}-\lambda R\mathbf{f})]+i[\phi(\mathbf{v})-\phi(\mathbf{v}-\lambda R\mathbf{f})]\}\rangle\\ =\exp\{\tfrac{1}{2}\langle[l(\mathbf{v})+l(\mathbf{v}-\lambda R\mathbf{f})-2\bar{l}]^2\rangle+2\bar{l}\\ -\tfrac{1}{2}\langle[\phi(\mathbf{v})-\phi(\mathbf{v}-\lambda R\mathbf{f})]^2\rangle\}\\ =\exp[C_l(0)+C_l(\lambda Rf)-2C_l(0)-\tfrac{1}{2}\mathfrak{D}_\phi(\lambda Rf)]\\ =\exp[-\tfrac{1}{2}\mathfrak{D}(\lambda Rf)],\end{aligned} \qquad (3.13)$$

where $\mathfrak{D}(r)$, defined as

$$\mathfrak{D}(r)=\mathfrak{D}_l(r)+\mathfrak{D}_\phi(r), \qquad (3.14)$$

is a quantity we call the "wave-structure function."[8] When we substitute (3.13) into (3.2), and define

$$\tau_0(f)=A^2B\int d\mathbf{v}W(\mathbf{v}-\lambda R\mathbf{f})W(\mathbf{v}), \qquad (3.15)$$

we get

$$\langle\tau(f)\rangle_{LE}=\tau_0(f)\exp[-\tfrac{1}{2}\mathfrak{D}(\lambda Rf)]. \qquad (3.16)$$

To obtain (3.16), we have used the fact that the exponential in (3.13) is independent of $\mathbf{v}$ so that it can be removed from the integrand of the $\mathbf{v}$ integration. Now, we need only note that the integral in (3.15) is simply the area of overlap of the two circles of diameter D whose centers are separated by a distance λRf. We choose B, whose value should be independent of the strength of turbulence,[9] to satisfy (2.3) when there is

[5] V. I. Tatarski, *Wave Propagation in a Turbulent Medium* (McGraw-Hill Book Co., New York, 1961), p. 209. This discusses only the distribution of l, but the same argument can be easily modified to apply to ϕ.

[6] Because atmospheric turbulence, or any refractive inhomogeneity can only redistribute radiant energy, not absorb it, for an infinite plane wave (or spherical wave), the average irradiance reaching any point must be independent of the strength of the turbulence. Otherwise, the average of the irradiance over a large collecting surface would not be a constant; but it must be, since all of the energy has to reach this surface. From Note 5, we see that the irradiance fluctuations are distributed in a log-normal manner. This and conservation of energy are compatible only if the center of the distribution, determined by $\bar{l}$, is related to the variance of the distribution $C_l(0)$. (Of course, when there is no turbulence, $\bar{l}$ and $C_l(0)$ are both zero.)

[7] D. M. Chase, J. Opt. Soc. Am. **56**, 33 (1966).

[8] The derivation of (3.13) with slightly different words would be sufficient to prove that $\exp[-\frac{1}{2}\mathfrak{D}(r)]$ is exactly equal to the mutual coherence function as used by Hufnagel and Stanley (cf. Ref. 10).

[9] The general approach of Sec. 3 to compute the long-exposure MTF has been used by E. A. Trabka, J. Opt. Soc. Am. **56**, 128 (1966), but by assuming that $\bar{l}=0$, he had to allow the normalization term, corresponding to B in the paper, to depend on the strength of the turbulence, which it should not.

no atmospheric effect. Thus we get

$$\tau_0(f)=\begin{cases}(2/\pi)[\cos^{-1}(\lambda Rf/D)-(\lambda Rf/D)(1-(\lambda Rf/D)^2)^{\frac{1}{2}}] \\ \qquad\qquad\qquad\qquad \text{if } \lambda Rf\leqslant D \\ 0 \qquad\qquad\qquad\qquad \text{if } \lambda Rf>D.\end{cases} \quad (3.17)$$

$\tau_0(f)$ is seen to be the MTF of a diffraction-limited lens.

If we wish, we can follow the lead of Hufnagel and Stanley[10] and consider $\langle\tau(f)\rangle_{LE}$ as the product of the lens MTF, $\tau_0(f)$, and a quantity $\exp[-\frac{1}{2}\mathfrak{D}(\lambda Rf)]$, which we can consider to be the atmosphere's MTF. This separation is meaningful since we can make the atmosphere's MTF manifestly independent of all lens parameters if we express our results in terms of angular resolution rather than linear resolution. We replace the frequency f (whose dimensions are cycles per unit length) with $f'=Rf$. (The dimensions of f' are cycles per radian-field-of-view.) Then the MTF of the atmosphere becomes $\exp[-\frac{1}{2}\mathfrak{D}(\lambda f')]$, which is manifestly independent of lens parameters. We shall see, however, that this procedure can not be extended to the short-exposure MTF. In that case, it is impossible to define an atmospheric MTF which is independent of lens parameters. For this reason, we prefer not to assign an MTF to the atmosphere.

IV. SHORT-EXPOSURE MTF

In this section, we are again concerned with the effects of random phase and log-amplitude fluctuations. Here, however, the exposure time used to form the images is so short that the part of the phase fluctuation which is associated with tilt of the isophase surface must be treated in a special manner (A rigorous analysis of the "shape" of a distorted wavefront, permitting us to identify a tilt, is presented in Ref. 2.) For a very-short exposure, tilt (as distinguished from warping of the isophase surface) does not affect the sharpness of a point image,[1] though it does produce a displacement of the image. It, therefore, does not play a part in determining MTF.

The MTF of a single short-exposure image is just as given in (2.8),

$$\tau(\mathbf{f})=A^2B\int d\mathbf{v}W(\mathbf{v}-\lambda R\mathbf{f})W(\mathbf{v})\exp\{[l(\mathbf{v})+l(\mathbf{v}-\lambda R\mathbf{f})] +i[\phi(\mathbf{v})-\phi(\mathbf{v}-\lambda R\mathbf{f})]\}, \quad (2.8)$$

which is to be distinguished from (3.1), in which, because of time averaging, an average of the exponential appears. To determine the average short-exposure MTF, $\langle\tau(\mathbf{f})\rangle_{SE}$, we must examine the measurement procedure and make our averaging process correspond to this procedure. In measuring the average transfer function for some image frequency $\mathbf{f}$, we do not add the harmonic component at frequency $\mathbf{f}$ from each of several images (i.e., we do not add the sine waves)—we add only the amplitudes of the components. To add the harmonic components themselves, we would need to know the phases of the components; but the phase is unknown since, to determine it, we would have to have an absolute position reference on each of the images. The lack of an absolute position reference on the image is equivalent to the unobservability of the tilt of the isophase surface and the displacement of the image. When we take the short-exposure average, we first suppress the effect of the tilt of the isophase surface on the harmonic components and then take the ensemble average.

Consider some instant of time for which the phase fluctuation of the wavefront is $\phi(\mathbf{v})$. Let $\mathbf{a}$ be a random vector related to $\phi(\mathbf{v})$ in such a manner than $\mathbf{a}\cdot\mathbf{v}$ gives the best fit to $\phi(\mathbf{v})$ in terms of a least-squares difference over the lens aperture, i.e.,

$$\frac{\partial}{\partial a_i}\int d\mathbf{v}W(\mathbf{v})[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]^2=0, \quad (4.1)$$

where a_i is a component of $\mathbf{a}$.

We now rewrite (2.8) as

$$\tau(\mathbf{f})=\exp(-i\mathbf{a}\cdot\lambda R\mathbf{f})A^2B\int d\mathbf{v}W(\mathbf{v}-\lambda R\mathbf{f})W(\mathbf{v}) \times\exp([l(\mathbf{v})+l(\mathbf{v}-\lambda R\mathbf{f})]+i\{[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}] -[\phi(\mathbf{v}-\lambda R\mathbf{f})-\mathbf{a}\cdot(\mathbf{v}-\lambda R\mathbf{f})]\}). \quad (4.2)$$

We now drop the unobservable tilt information contained in the factor $\exp(-i\mathbf{a}\cdot\lambda R\mathbf{f})$, so that

$$\hat{\tau}(\mathbf{f})=A^2B\int d\mathbf{v}W(\mathbf{v}-\lambda R\mathbf{f})W(\mathbf{v}) \times\exp([l(\mathbf{v})+l(\mathbf{v}-\lambda R\mathbf{f})]+i\{[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}] -[\phi(\mathbf{v}-\lambda R\mathbf{f})-\mathbf{a}\cdot(\mathbf{v}-\lambda R\mathbf{f})]\}). \quad (4.3)$$

$\hat{\tau}(\mathbf{f})$ is the quantity we actually measure when we examine a single image without an absolute reference point. The ensemble-average short-exposure MTF, $\langle\tau(\mathbf{f})\rangle_{SE}$, is

$$\langle\tau(\mathbf{f})\rangle_{SE}=\langle(\mathbf{f}\hat{\tau})\rangle. \quad (4.4)$$

To take the ensemble average of the exponential in (4.3), we have to make the following three assumptions.

(I) the distribution of $\mathbf{a}$, like that of ϕ and l, is gaussian; (II) the distribution of $[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]$ is independent of the distribution of $\mathbf{a}$; and (III) the distribution of $\{[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]-[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]\}$ is independent of the distribution of $[l(\mathbf{v})+l(\mathbf{v}')]$.

Assumption (I) can be justified on the basis that $\mathbf{a}$ is generated in a linear manner by $\phi(\mathbf{v})$, which is a gaussian random variable. The accuracy of assumption (II) can be argued on the basis of symmetry: For any given tilt, as measured by $\mathbf{a}$, the probability of a given

[10] R. E. Hufnagel and N. R. Stanley, J. Opt. Soc. Am. 54, 52 (1964).

isophase surface warping relative to the tilt surface, as given by $[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]$, and the probability of the mirror image warping about the tilt surface, as given by $-[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]$, are equal. Consequently, $\langle[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]\times\mathbf{a}\cdot\mathbf{v}\rangle$ vanishes. This, together with assumption (I), justifies assumption (II).[11] Assumption (III) can be justified on the basis of (3.3) and the following consideration. For a given log-amplitude fluctuation $l(\mathbf{v})$, some particular tilt $\mathbf{a}$ and its mirror image $-\mathbf{a}$ are equally likely. Hence $\langle l(\mathbf{v})\mathbf{a}\cdot\mathbf{v}\rangle$ must vanish. The fact that $\langle l(\mathbf{v})\mathbf{a}\cdot\mathbf{v}\rangle=0$ and (3.3) are sufficient to prove assumption (III).

From assumptions (I) and (III) and by the same manipulations as were used to treat the log-amplitude variation in (3.13), we conclude that

$$\begin{aligned}\langle\exp([l(\mathbf{v})+l(\mathbf{v}-\lambda R\mathbf{f})]+i\{[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]\\-[\phi(\mathbf{v}-\lambda R\mathbf{f})-\mathbf{a}(\mathbf{v}-\lambda R\mathbf{f})]\})\rangle\\=\exp(-\tfrac{1}{2}\mathfrak{D}_l(\lambda Rf)-\tfrac{1}{2}\langle\{[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]\\-[\phi(\mathbf{v}-\lambda R\mathbf{f})-\mathbf{a}\cdot(\mathbf{v}-\lambda R\mathbf{f})]\}^2\rangle).\end{aligned}\quad(4.5)$$

By simple manipulation, we can show that

$$\begin{aligned}\{[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]-[\phi(\mathbf{v}-\lambda R\mathbf{f})-\mathbf{a}\cdot(\mathbf{v}-\lambda R\mathbf{f})]\}^2\\=[\phi(\mathbf{v})-\phi(\mathbf{v}-\lambda R\mathbf{f})]^2-(\mathbf{a}\cdot\lambda R\mathbf{f})^2\\+2\{[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]-[\phi(\mathbf{v}\lambda R\mathbf{f})\\-\mathbf{a}\cdot(\mathbf{v}-\lambda R\mathbf{f})]\}\mathbf{a}\cdot\lambda R\mathbf{f}.\end{aligned}\quad(4.6)$$

Assumption (II) implies that we can drop the curly-bracket term on the right-hand side of (4.6) when we take the ensemble average. Thus,

$$\begin{aligned}\langle\exp([l(\mathbf{v})+l(\mathbf{v}-\lambda R\mathbf{f})]+i\{[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]\\-[\phi(\mathbf{v}-\lambda R\mathbf{f})-\mathbf{a}\cdot(\mathbf{v}-\lambda R\mathbf{f})]\})\rangle\\=\exp\{-\tfrac{1}{2}\mathfrak{D}_l(\lambda Rf)-\tfrac{1}{2}\mathfrak{D}_\phi(\lambda Rf)+\tfrac{1}{2}\langle(\mathbf{a}\cdot\lambda R\mathbf{f})^2\rangle\},\end{aligned}\quad(4.7)$$

so that

$$\langle\tau(\mathbf{f})\rangle_{SE}=A^2B\int d\mathbf{v}W(\mathbf{v}-\lambda R\mathbf{f})W(\mathbf{v})\times\exp\{-\tfrac{1}{2}[\mathfrak{D}(\lambda Rf)-\langle(\mathbf{a}\cdot\lambda R\mathbf{f})^2\rangle]\}.\quad(4.8)$$

The exponential is independent of the variable of integration and so can be removed from the integrand. When we carry out the integration and normalization as before, we get

$$\langle\tau(\mathbf{f})\rangle_{SE}=\tau_0(f)\exp\{-\tfrac{1}{2}[\mathfrak{D}(\lambda Rf)-\langle(\mathbf{a}\cdot\lambda R\mathbf{f})^2\rangle]\}.\quad(4.9)$$

The quantity $\langle(\mathbf{a}\cdot\lambda R\mathbf{f})^2\rangle$ can be evaluated by equating $\mathbf{a}\cdot\mathbf{v}$ with $a_2F_2(\mathbf{v})+a_3F_3(\mathbf{v})$ in Ref. 2. We see that

$$\mathbf{a}\cdot\mathbf{a}=(64/\pi D^4)[(a_2)^2+a_3)^2].\quad(4.10)$$

From Eq. (5.8a) of Ref. 2, we find that

$$\langle\mathbf{a}\cdot\mathbf{a}\rangle=\frac{64}{D^4}\int_0^D rdr[\mathfrak{F}_C(r,D)-\mathfrak{F}_L(r,D)]\mathfrak{D}_\phi(r),\quad(4.11)$$

where

$$\mathfrak{F}_C(r,D)=(\pi)^{-1}[2\cos^{-1}(r/D)-2(r/D)(1-(r/D)^2)^{\frac{1}{2}}],\quad(4.12a)$$

$$\mathfrak{F}_L(r,D)=(\pi)^{-1}\{6\cos^{-1}(r/D)-[14(r/D)-8(r/D)^3](1-(r/D)^2)^{\frac{1}{2}}\}.\quad(4.12b)$$

Since $\mathbf{a}$ is an isotropically distributed random vector and $\mathbf{f}$ a nonrandom vector

$$\langle(\mathbf{a}\lambda R\mathbf{f})^2\rangle=\tfrac{1}{2}(\lambda Rf)^2\langle\mathbf{a}\cdot\mathbf{a}\rangle.\quad(4.13)$$

If we substitute (4.11) into (4.13) and that into (4.9), we have the evaluation of $\langle\tau(f)\rangle_{SE}$, the short-exposure MTF.

V. STRUCTURE FUNCTIONS

There are good theoretical and experimental reasons[12–14] to believe that the wave-structure function, $\mathfrak{D}(r)$ can be written as

$$\mathfrak{D}(r)=\mathcal{A}r^{5/3},\quad(5.1)$$

where $\mathcal{A}$ is a constant whose value depends on the propagation-path length, the wavelength, the "strength" of the turbulence along the path, and the nature of the unperturbed wavefront. The theory, it should be noted, is based on the Rytov approximation to solve the wave equation and the Kolmogoroff hypothesis to provide the statistics of atmospheric turbulence.[15] (A discussion of the calculation of $\mathcal{A}$, when the unperturbed wavefront is an infinite plane wave is given elsewhere.[16]) It is convenient to define a quantity r_0 such that

$$r_0\simeq(6.88/\mathcal{A})^{3/5},\quad(5.2)$$

in terms of which

$$\mathfrak{D}(r)=6.88(r/r_0)^{5/3}.\quad(5.3)$$

(The significance of the factor 6.88, which is more precisely given by $2\{(24/5)\Gamma(6/5)\}^{5/6}$, is contained in

[11] Dr. G. R. Heidbreder, of Aerospace Corp., has informed me that he has been able to show that, at least for a one-dimensional aperture, assumption II must be viewed as an approximation, and has found a weak correlation between $\phi(v)-av$ and a. As he has shown, however, the inaccuracy in assumption II drops out in the v-integration when assumption II is used to obtain a result like (4.8), providing that $\frac{1}{2}\{[\phi(\mathbf{v})-\mathbf{a}\cdot\mathbf{v}]-[\phi(\mathbf{v}-\lambda R\mathbf{f})-\mathbf{a}\cdot(\mathbf{v}-\lambda R\mathbf{f})]\}^2$ is small enough that the approximation $e^x\simeq1+x$ can be made. Whenever the short-exposure MTF is not too severely degraded by atmospheric turbulence, this smallness condition is satisfied.

[12] D. L. Fried and J. D. Cloud, "Optical Propagation in the Atmosphere: Theoretical Evaluation and Experimental Determination of the Phase Structure Function," presented at the Conference on Atmospheric Limitations to Optical Propagation at the U. S. National Bureau of Standards, CRPL, 18–19 March 1965.

[13] V. I. Tatarski, Ref. 5, Eq. (8.20).

[14] Though previously published results prove the validity of (5.1) for only an infinite plane wave, it can be shown that (5.1) is equally applicable to the propagation of a spherical wave. The coefficient $\mathcal{A}$ is different. For horizontal propagation, it can be shown that the spherical-wave coefficient is exactly 3/8 of the coefficient for an infinite plane wave propagating over the same path.

[15] The Rytov approximation was introduced and so attributed by Tatarski, Ref. 5, p. 269, though we have been unable to find such an approximation discussed in Rytov's rather lengthy "source" paper.

[16] D. L. Fried, Ref. 2. Appendix C.

the fact that it makes the knee of curve A in Fig. 1 occur at $D=r_0$.)

It can be shown[17] that the phase-structure function $\mathfrak{D}_\phi(r)$ can be written as

$$\mathfrak{D}_\phi(r)=\alpha(r)\mathfrak{D}(r), \tag{5.4}$$

where $\alpha(r)$ is a function which varies from unity for $r\gg(L\lambda)^{\frac{1}{2}}$ to one-half for $r\ll(L\lambda)^{\frac{1}{2}}$. L is the length of the propagation path through the turbulent medium. We restrict our attention to the two extreme cases, which we refer to as the near-field case, for which nearly all significant values of r satisfy the inequality $r\gg(L\lambda)^{\frac{1}{2}}$, and the far-field case, for which nearly all significant values of r satisfy the inequality $r\ll(L\lambda)^{\frac{1}{2}}$. In the problem we are studying in this paper, we may replace the conditions on r by conditions on the lens diameter, D, i.e., $D\gg(L\lambda)^{\frac{1}{2}}$ (near field) and $D\ll(L\lambda)^{\frac{1}{2}}$ (far field). Thus

$$\mathfrak{D}_\phi(r)\simeq\mathfrak{D}(r) \quad \text{(near field)}, \tag{5.5a}$$

$$\mathfrak{D}_\phi(r)\simeq\tfrac{1}{2}\mathfrak{D}(r) \quad \text{(far field)}. \tag{5.5b}$$

We are now in a position to evaluate the long-exposure MTF, $\langle\tau(f)\rangle_{LE}$, and the short-exposure MTF for the near-field and far-field cases, which we denote by ${}_{nf}\langle\tau(f)\rangle_{SE}$ and ${}_{ff}\langle\tau(f)\rangle_{SE}$, respectively. (For the long-exposure MTF, there is no distinction between the near-field and far-field cases.) After we substitute (5.3) into (3.16), we see that

$$\langle\tau(f)\rangle_{LE}=\tau_0(f)\exp[-3.44(\lambda Rf/r_0)^{5/3}]. \tag{5.6}$$

If we substitute (5.5a or b) into (4.11), and utilize the fact that

$$\int_0^1 u^{8/3}\mathfrak{F}_C(u,1)du\simeq3.68\times10^{-2}, \tag{5.7a}$$

$$\int_0^1 u^{8/3}\mathfrak{F}_L(u,1)du\simeq4.73\times10^{-2}, \tag{5.7b}$$

we get

$$\langle(\mathbf{a}\cdot\lambda R\mathbf{f})^2\rangle=\begin{cases}1.026\times6.88(\lambda Rf/r_0)^{5/3}(\lambda Rf/D)^{\frac{1}{3}} & \text{(near field)}\\ (1.026/2)\times6.88(\lambda Rf/r_0)^{5/3}(\lambda Rf/D)^{\frac{1}{3}} & \text{(far field).}\end{cases} \tag{5.8}$$

Thus, we obtain

$${}_{nf}\langle\tau(f)\rangle_{SE}\simeq\tau_0(f)\exp\{-3.44(\lambda Rf/r_0)^{5/3}\times[1-(\lambda Rf/D)^{\frac{1}{3}}]\}, \tag{5.9a}$$

$${}_{ff}\langle\tau(f)\rangle_{SE}\simeq\tau_0(f)\exp\{-3.44(\lambda Rf/r_0)^{5/3}\times[1-\tfrac{1}{2}(\lambda Rf/D)^{\frac{1}{3}}]\}, \tag{5.9b}$$

where we have approximated 1.026 by unity in (5.9a and b).[18]

If we compare (5.9a and b) with (5.6), the effect of using a short exposure rather than a long exposure becomes obvious. As the value of f approaches $D/\lambda R$, which is the cut-off frequency of $\tau_0(f)$, the effect of using a short exposure becomes more and more important. Because we take the cube root of f over the cut-off frequency, the ratio does not have to be very close to unity for the improvement to be substantial. This is especially true in the near-field case, where we can recover almost all of the highest-frequency response.

An interesting measure of the performance of an imaging system is provided by the quantity $\mathfrak{R}$, which we refer to as the resolution, and which we define as the integral over spatial frequencies of the system's ensemble-average MTF.

$$\mathfrak{R}=\int d\mathbf{f}\langle\tau(\mathbf{f})\rangle \tag{5.10}$$

We may think of $\mathfrak{R}$ as one of the three standard measures of image quality,[19] as proportional to the Strehl definition of the optical system, or even, in terms with which an electrical engineer would be more familiar, as the bandwidth of the system. We do not attempt to justify use of this particular measure of image quality rather than another, as the question of the best measure of image quality is still unsettled. The same procedure that we use in evaluating $\mathfrak{R}$ can be applied equally well to the other measures of image quality, if desired.

In anticipation of a more general treatment of the effect of exposure time, we subscript $\mathfrak{R}$ with ∞ to denote the long-exposure resolution and with a 0 for the short-exposure resolution. To indicate resolution in the near-field and far-field, where necessary we precede the $\mathfrak{R}$ with a subscript nf or ff, respectively. From (3.17), (5.6), and (5.10) we get

$$\mathfrak{R}_\infty=4\frac{D^2}{\lambda R}\int_0^1 udu[\cos^{-1}u-u(1-u^2)^{\frac{1}{2}}]\times\exp\left[-3.44\left(\frac{D}{r_0}\right)^{5/3}u^{5/3}\right]. \tag{5.11}$$

If we use (5.9a or b) instead of (5.6), we get

$${}_{nf}\mathfrak{R}_0=4\frac{D^2}{\lambda R}\int_0^1 udu[\cos^{-1}u-u(1-u^2)^{\frac{1}{2}}]\times\exp\left[-3.44\left(\frac{D}{r_0}\right)^{5/3}u^{5/3}(1-u^{\frac{1}{3}})\right], \tag{5.12a}$$

[17] An example of this for the infinite plane-wave propagation case is provided by Tatarski, Ref. 5, Eqs. (8.20), (8.21), and (8.22).

[18] Dr. R. E. Hufnagel has informed me that he has arrived at a result equivalent to (5.9a), through an analysis differing in substantial features from that presented here.

[19] E. L. O'Neill, *Introduction to Statistical Optics* (Addison-Wesley Publishing Co., Reading, Massachusetts, 1963), p. 106.

TABLE I. Dependence of normalized resolution on normalized diameter.

D/r_0	$\mathcal{R}_\infty/\mathcal{R}_{max}$	${}_{ff}\mathcal{R}_0/\mathcal{R}_{max}$	${}_{nf}\mathcal{R}_0/\mathcal{R}_{max}$
0.1	0.00978	0.00988	0.00997
0.5	0.1852	0.208	0.237
1.0	0.445	0.586	0.844
2.0	0.699	1.048	2.36
3.0	0.797	1.202	3.32
3.5	0.826	1.217	3.49
3.8	0.837	1.225	3.50
4.0	0.848	1.234	3.48
5.0	0.878	1.249	3.20
7.0	0.913	1.253	2.52
10	0.939	1.242	2.05
15	0.960	1.222	1.780
20	0.970	1.206	1.654
30	0.980	1.183	1.524
50		1.156	1.407
100		1.124	1.298
200		1.098	1.223
500			1.156
1000			1.120

$$
{}_{ff}\mathcal{R}_0 = 4\frac{D^2}{\lambda R}\int_0^1 u\,du[\cos^{-1}u - u(1-u^2)^{\frac{1}{2}}] \times \exp\left[-3.44\left(\frac{D}{r_0}\right)^{5/3} u^{5/3}(1-\tfrac{1}{2}u^{\frac{1}{3}})\right]. \quad (5.12b)
$$

It is interesting to consider $\mathcal{R}_{max}$, the limiting value of $\mathcal{R}_\infty$ as the lens diameter becomes arbitrarily large. We refer to this as the "limiting resolution."

$$
\mathcal{R}_{max} = \lim_{D\to\infty} \mathcal{R}_\infty. \quad (5.13)
$$

When we carry out the evaluation of (5.11) in this limit, we get

$$
\mathcal{R}_{max} = (\pi/4)(r_0/\lambda R)^2. \quad (5.14)
$$

This has the dimensions of cycles squared per unit area

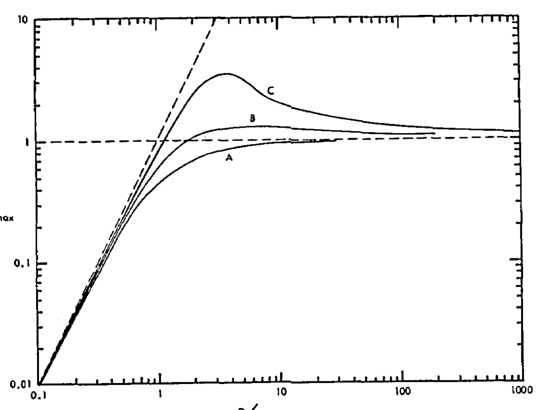

FIG. 1. Dependence of normalized resolution, $\mathcal{R}/\mathcal{R}_{max}$, on normalized lens diameter, D/r_0. Curve A—long exposure results, $\mathcal{R}_\infty/\mathcal{R}_{max}$. Curve B—Short exposure far-field results, ${}_{ff}\mathcal{R}_0/\mathcal{R}_{max}$. Curve C—Short exposure near-field results, ${}_{nf}\mathcal{R}_0/\mathcal{R}_{max}$.

(in the focal plane). If we normalize $\mathcal{R}_\infty$, ${}_{nf}\mathcal{R}_0$, and ${}_{ff}\mathcal{R}_0$ by dividing each by the "limiting resolution," we get

$$
\frac{\mathcal{R}_\infty}{\mathcal{R}_{max}} = \frac{16}{\pi}\left(\frac{D}{r_0}\right)^2 \int_0^1 u\,du[\cos^{-1}u - u(1-u^2)^{\frac{1}{2}}] \times \exp\left[-3.44\left(\frac{D}{r_0}\right)^{5/3} u^{5/3}\right], \quad (5.15)
$$

$$
\frac{{}_{nf}\mathcal{R}_0}{\mathcal{R}_{max}} = \frac{16}{\pi}\left(\frac{D}{r_0}\right)^2 \int_0^1 u\,du[\cos^{-1}u - u(1-u^2)^{\frac{1}{2}}] \times \exp\left[-3.44\left(\frac{D}{r_0}\right) u^{5/3}(1-u^{\frac{1}{3}})\right], \quad (5.16a)
$$

$$
\frac{{}_{ff}\mathcal{R}_0}{\mathcal{R}_{max}} = \frac{16}{\pi}\left(\frac{D}{r_0}\right)^2 \int_0^1 u\,du[\cos^{-1}u - u(1-u^2)^{\frac{1}{2}}] \times \exp\left[-3.44\left(\frac{D}{r_0}\right) u^{5/3}(1-\tfrac{1}{2}u^{\frac{1}{3}})\right]. \quad (5.16b)
$$

These three integrals have been evaluated numerically for a range of values of D/r_0 and are plotted in Fig. 1. Table I gives the values from which the graphs were drawn.

VI. DISCUSSION OF RESULTS

Perhaps the first point to be commented on should be the exact agreement between the long-exposure MTF, given by (5.6) of this paper and the equivalent quantity given by (2.5), (7.2), and (7.6) of the paper of Hufnagel and Stanley,[10] referred to in this discussion as HS. This exact agreement may provide some insight into some questions of accuracy that have been raised recently. HS have questioned the accuracy of the Rytov approximation[15] used by Tatarski to obtain (5.1). More recently, Chase[20] has pointed out that HS inadvertently suppressed what appears to be a small but unevaluated term in the equation they derived and solved for the propagation of mutual coherence; i.e., HS have an unevaluated approximation in their results. The HS and Rytov approximations are so fundamentally different that, before seeing the solutions they lead to, no one could have reason to expect that they would result in the same solution—unless the approximations are sufficiently accurate that they both yield the exact solution. In view of the apparent fundamental difference between the two approximations, and in the absence of any argument to indicate that they might be related (except, of course, the argument that they lead to the same result), the agreement of the results can be taken as bolstering confidence in the accuracy of the approximations. This obviously is only a heuristic argument for the accuracy of the approximations.

[20] D. M. Chase, J. Opt. Soc. Am. 55, 1559 (1965).

As a second point of interest, the existence of the limiting resolution, $\mathcal{R}_{max}$ implies that atmospheric turbulence places an absolute upper limit on the resolution that can be obtained with a long exposure through the atmosphere. This resolution, given in (5.14) is exactly the same resolution that would be obtained in the absence of the atmosphere with a diffraction-limited lens of diameter r_0. If we examine the source of (5.1), we find that A is inversely proportional to the square of λ, so r_0 is directly proportional to the six-fifths power of λ. From (5.14) we see that this means that the limiting resolution should increase as the two-fifth power of wavelength. At 16 μ, the limiting resolution (in two-dimensions) should be four-times that at one-half micron, i.e., the angular spread of an image formed at 16 μ by very large optics should be one-half the angular spread at one-half micron, if atmospheric turbulence is the limiting factor.

The fact that significantly better resolution may be obtainable with a short exposure than with a long exposure is obvious from Fig. 1 and hardly needs elaboration here. The point to be noted is the significance of this fact in terms of the design of optical-imagery experiments for studying atmospheric effects and propagation theories. Because it is difficult to make an exact statement as to whether an experiment was carried out under near-field or far-field conditions, it is desirable that imaging experiments for studying propagation theory be designed to apply to the long-exposure theory. (This consideration may be relaxed when more comprehensive theoretical results become available.) Because of practical considerations, many imaging experiments have been performed in which the short-exposure theory appears to be more applicable than the long-exposure theory.[21–23] In fact, HS comment that the disagreement of the data of Djurle and Bäck[21] with their (modified) theory is probably due to the exposure time being too short. We observe that the disagreement of theory and experiment as given in Fig. 8 of HS is most pronounced at the higher spatial frequencies, with the experimental data indicating a larger MTF than HS's long-exposure theory. This is in qualitative agreement with the results developed here.

It is interesting to compare the results contained in Fig. 1 with the semiquantitative predictions comparing long- and short-exposure resolution in Ref. 2. In that document it was estimated that the short-exposure near-field (treated there as an assumption of no significant intensity fluctuation) resolution would approach its maximum value at a lens diameter of 3.4 r_0, while the long-exposure resolution would approach its maximum value at a lens diameter of r_0. The short-exposure resolution was expected to be $(3.4)^2 \simeq 11.4$ times as much as the long-exposure resolution. Examining Fig. 1, we see that the estimated diameters of 3.4 r_0 and r_0 were reasonably accurate and that if we compare the short-exposure resolutions at diameters 3.4 r_0, and the long-exposure resolution at diameter r_0, we get a factor of 7.8, which is in reasonable agreement with the predictions. However, the prediction is seen to be misleading in-as-much as the long-exposure resolution increases significantly as the diameter increases from r_0 to 3.4 r_0. For long- and short-exposures, both taken with a diameter of 3.8 r_0 (where the peak short-exposure resolution occurs), the ratio of resolutions is only 4.3, which is significant, but significantly less than the 11.4 estimated.

Finally, it is worth commenting on the fact that the expression given for $\mathcal{R}$ can be obtained in exactly the same form, for the relative effect of atmospheric wavefront distortion on the average signal-to-noise ratio obtainable in an optical-heterodyne receiver. The short-exposure results correspond to the case in which the orientation of the local-oscillator wavefront is made to track perfectly the instantaneous average tilt of the wavefront of the collected-signal. The long-exposure results correspond to the more conventional concept of an optical-heterodyne receiver in which no tracking is performed by the local oscillator. The results of Fig. 1 are thus applicable. It is interesting to note that Chase[7] has obtained a factor of 3.4 increase in useful collector diameter in going from a nontracking to a tracking optical-heterodyne receiver. Because of the nature of the approximations made, he concluded that there should be an improvement in signal-to-noise ratio of about 11.5. In the previous paragraph, we have discussed these factors and their applicability. The same comments seem called for here. To sum them up, with a collector size selected to be as large as useful, the change from a nontracking to a tracking heterodyne receiver achieves about a factor-of-four improvement in signal-to-noise (power) ratio, i.e., 6 dB. This of course applies in the near-field. In the far-field, the improvement is almost negligible.

[21] E. Djurle and A. Bäck, J. Opt. Soc. Am. 51, 1029 (1961).
[22] C. E. Coulman, J. Opt. Soc. Am. 55, 806 (1965).
[23] C. B. Rogers, J. Opt. Soc. Am. 55, 1151 (1965).